

ORDERINGS OF WITZEL-ZAREMSKY-THOMPSON GROUPS

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ABSTRACT. We prove the orderability of the Witzel-Zaremsky-Thompson group for a direct system of orderable groups under a certain compatibility assumption.

1. INTRODUCTION

Thompson's groups are interesting countable groups and several kinds of their generalizations have been developed and studied. As one of such examples, Witzel and Zaremsky introduced the group $\mathcal{T}(G_*)$ associated to a direct system (G_n) of groups satisfying certain axioms called the cloning system [7]. We call it the Witzel-Zaremsky-Thompson group for (G_n) .

The Witzel-Zaremsky-Thompson group $\mathcal{T}(G_*)$ can be considered as not only a generalization of Thompson's groups but also a limit of the groups G_n . We prove the group $\mathcal{T}(G_*)$ inherits the orderability of the groups G_n under the assumption that there exists an ordering of G_n preserved by the cloning maps of which we give the definition in Section 2. For bi-orderability, this assumption is a necessary condition. The following is the main proposition of this article.

Proposition 1.1. *Let (G_n) be a cloning system.*

- (i) *The Witzel-Zaremsky-Thompson group $\mathcal{T}(G_*)$ for (G_n) is left-orderable if the groups G_n admit left-orderings which are preserved by the cloning maps.*
- (ii) *The group $\mathcal{T}(G_*)$ is bi-orderable if and only if the groups G_n admit bi-orderings which are preserved by the cloning maps.*

The “if parts” of Proposition 1.1 is proved by the argument which is a straightforward generalization of that used in [3]. Although it seems to be emphasized in [3] that the cloning system is assumed to be so-called pure and thus the cloning maps are group homomorphisms, it is not an essential assumption.

2. PRELIMINARIES

Let (G_n) be a direct system of groups. That is, an injective homomorphism $G_n \hookrightarrow G_{n+1}$ is given for each $n \in \mathbb{N}$. The Witzel-Zaremsky-Thompson group $\mathcal{T}(G_*)$ is simply a certain subgroup of the Brin-Zappa-Szép product $G \bowtie \mathcal{F}$, where G is the direct limit of the system (G_n) and \mathcal{F} is the monoid of binary forests. However we recall the definition of $\mathcal{T}(G_*)$ in a fashion describing its elements.

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Let \mathfrak{S}_n be the symmetric group on the set $\{1, \dots, n\}$. Set the map $\varsigma_n^k: \mathfrak{S}_n \rightarrow \mathfrak{S}_{n+1}$ by

$$\varsigma_n^k(g)m = \begin{cases} gm & \text{if } m \leq k, gm \leq gk, \\ gm + 1 & \text{if } m < k, gm > gk, \\ g(m-1) & \text{if } m > k, g(m-1) < gk, \\ g(m-1) + 1 & \text{if } m > k, g(m-1) \geq gk. \end{cases}$$

Definition 2.1. Suppose that a group homomorphism $\rho_n: G_n \rightarrow \mathfrak{S}_n$ and a map $\kappa_n^k: G_n \rightarrow G_{n+1}$ is given for each $n \in \mathbb{N}$ and for each $k = 1, \dots, n$.

The triple $(G_n) = ((G_n), (\rho_n), (\kappa_n^k))$ is a *cloning system* if the following three axioms are satisfied.

- (1) $\kappa_n^k(gh) = \kappa_n^k(g)\kappa_n^{\rho_n(g)k}(h)$ for $g, h \in G_n$,
- (2) $\kappa_{n+1}^k \circ \kappa_n^l = \kappa_{n+1}^{l+1} \circ \kappa_n^k$ for $1 \leq k < l \leq n$, and
- (3) $\rho_n(\kappa_n^k(g)) = \varsigma_n^k(\rho_n(g))$ or $s_k \varsigma_n^k(\rho_n(g))$ for $g \in G_n$ and for $1 \leq k \leq n$, where the symbol s_k means the transposition $(k, k+1) \in \mathfrak{S}_{n+1}$.

The maps $\kappa_n^k: G_n \rightarrow G_{n+1}$ are called the *cloning maps* of the cloning system (G_n) .

Suppose that a cloning system $(G_n) = ((G_n), (\rho_n), (\kappa_n^k))$ is given. A tree diagram is a triple (T_-, g, T_+) , where $g \in G_n$ for certain $n \in \mathbb{N}$ and T_\pm are rooted planar binary trees with n leaves. A *simple expansion* of (T_-, g, T_+) is a tree diagram of the form $(T'_-, \kappa_n^k(g), T'_+)$. Here, T'_- and T'_+ are trees obtained by adjoining carets to the k -th and $(\rho(g)k)$ -th leaves of T_- and T_+ , respectively. An *expansion* of (T_-, g, T_+) is a tree diagram obtained as an iterated simple expansions. Two tree diagrams are *equivalent* if they have a common expansion. The equivalence class represented by a tree diagram (T_-, g, T_+) will be denoted by $[T_-, g, T_+]$. The *Witzel-Zaremsky-Thompson group* $\mathcal{T}(G_*)$ consists of equivalence classes of tree diagrams as a set. For two elements $[S_-, f, S_+], [T_-, g, T_+] \in \mathcal{T}(G_*)$, their product is defined as follows. There always exist tree diagrams (S'_-, f', S'_+) and (T'_-, g', T'_+) representing $[S_-, f, S_+]$ and $[T_-, g, T_+]$, respectively and such that $S'_+ = T'_-$. The product $[S_-, f, S_+][T_-, g, T_+]$ is defined to be $[S'_-, f'g', T'_+]$. This is well-defined.

When we fix a left-ordering or a bi-ordering $<_n$ of G_n for each n , we say the cloning maps κ_n^k 's *preserve the orderings* if $1 <_{n+1} \kappa_n^k(g)$ for any n and any k whenever $1 <_n g$.

3. PROOF OF THE MAIN PROPOSITION

Proof of Proposition 1.1. Let $(G_n) = ((G_n), (\rho_n), (\kappa_n^k))$ be a cloning system. Suppose that a left-ordering $<_n$ of the group G_n is given for each n and the orderings $<_n$ are preserved by the cloning maps. Note that the subgroup of $\mathcal{T}(G_*)$ consisting of elements represented by the form $(T_-, 1, T_+)$ is isomorphic to the Thompson's group F . We denote this subgroup again by F . Since the Thompson's group F is known to be bi-orderable, we fix a left-ordering $<_F$ of F . Set the subset Π of the Witzel-Zaremsky-Thompson group $\mathcal{T}(G_*)$ for (G_n) by

$$\Pi = \{[T_-, g, T_+] \in \mathcal{T}(G_*); g \in G_n, 1 <_n g\} \amalg \{f \in F; 1 <_F f\}.$$

Since the orders $<_n$ are preserved by the cloning maps, the set Π is well-defined. Further it is easy to verify that Π is a positive cone and thus defines a left-ordering of $\mathcal{T}(G_*)$. This completes the proof of the statement (i).

The "if part" of the statement (ii) follows similarly to (i). In fact, if we assume further that the orders $<_n$ and $<_F$ are bi-invariant, then the positive cone Π defined

above is conjugation-invariant and thus the obtained left-ordering on the group $\mathcal{T}(G_*)$ is also bi-invariant.

Now we prove the converse. Let $<$ be a bi-ordering of $\mathcal{T}(G_*)$. For $n \in \mathbb{N}$, take a rooted binary tree T with n leaves. The subgroup consisting of the elements of $\mathcal{T}(G_*)$ of the form $[T, g, T]$ where $g \in G_n$ is isomorphic to the group G_n and a bi-ordering $<_n$ of G_n is induced by $<$. Since $[S, g, S] = [S, 1, T][T, g, T][T, 1, S]$ for any other rooted binary tree S with n leaves and for any $g \in G_n$, the bi-ordering $<_n$ is independent of the choice of the tree T . Furthermore, for any rooted binary trees T and S with n leaves, $1 < [T, g, S]$ if and only if $1 <_n g$. This fact is easily verified since either $[T, 1, S]$ or $[S, 1, T]$ is positive. Thus the bi-orderings $<_n$ of G_n 's are preserved by the cloning maps. \square

For a group G , we denote by $BO(G)$ the space consisting of bi-orderings of G . By the proof of Proposition 1.1 (ii), we have the following.

Corollary 3.1. *The space $BO(\mathcal{T}(G_*))$ is homeomorphic to the direct product of $BO(F)$ and the space of families of bi-orderings of (G_n) preserved by the cloning maps.*

The space $BO(F)$ is known to be isomorphic to the disjoint union of four copies of the Cantor set and eight isolated points [5].

4. EXAMPLES

In this section we introduce examples of Witzel-Zaremsky-Thompson groups to which Proposition 1.1 is applicable.

4.1. The braided Thompson group. Let B_n be the braid group of n strands and $\sigma_1, \dots, \sigma_{n-1} \in B_n$ the Artin generators. Taking $\rho_n: B_n \rightarrow \mathfrak{S}_n$ to be the natural projection and setting $\kappa_n^k: B_n \rightarrow B_{n+1}$ by

$$\kappa_n^k(\sigma_i) = \begin{cases} \sigma_{i+1} & \text{if } k < i, \\ \sigma_{i+1}\sigma_i & \text{if } k = i, \\ \sigma_i\sigma_{i+1} & \text{if } k = i + 1, \\ \sigma_i & \text{if } i + 1 < k, \end{cases}$$

we have a well-defined cloning system on (B_n) . Then the Witzel-Zaremsky-Thompson group $\mathcal{T}(B_n)$ for (B_n) is isomorphic to the group called the braided Thompson group BV which was independently introduced by Brin and Dehornoy [2][4].

Recall that $\beta \in B_n$ is positive with respect to the Dehornoy ordering if and only if β is represented by a word w in the Artin generators which satisfies the following condition (D):

(D) σ_i occurs in w but $\sigma_1^{\pm 1}, \dots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$ does not for certain i .

It is not difficult to verify only by definition of the cloning maps that if w is a representation of β satisfying the condition (D) then $\kappa_n^k(w)$ also. Hence the following lemma holds:

Lemma 4.1. *The Dehornoy orderings of the braid groups B_n are preserved by the cloning maps of (B_n) defined above.*

By Proposition 1.1 and Lemma 4.1, we have in easy way the following theorem which was first proved in [4].

Theorem 4.2. *The braided Thompson group BV is left-orderable.*

4.2. The pure braided Thompson group. Let P_n be the pure braid group of n strands. We also denote the restriction on P_n of the map $\kappa_n^k: B_n \rightarrow \mathfrak{S}_n$ we set in the previous subsection again by κ_n^k . Take $\rho_n: P_n \rightarrow \mathfrak{S}_n$ to be the trivial homomorphism. Then we have a well-defined cloning system on (P_n) and the Witzel-Zaremsky-Thompson group $\mathcal{T}(P_n)$ for (P_n) which is isomorphic to the pure braided Thompson group BF introduced by Brady, Burillo, Cleary and Stein in [1].

Lemma 4.3 ([3]). *The orderings on the braid groups P_n induced from the Magnus orderings of the free groups via the Artin combing are preserved by the cloning maps on (P_n) defined above.*

Theorem 4.4 ([3]). *The pure braided Thompson group BF is bi-orderable.*

4.3. Witzel-Zaremsky-Thompson group for direct powers of a group. For arbitrary group G , set G_n to be the n -th direct power G^n of G . Fix injective homomorphisms ϕ_1, ϕ_2 of G . If we define $\kappa_n^k: G^n \rightarrow G^{n+1}$ by $\kappa_n^k(g_1, \dots, g_n) = (g_1, \dots, \phi_1(g_k), \phi_2(g_k), \dots, g_n)$ and set $\rho_n: G^n \rightarrow \mathfrak{S}_n$ to be the trivial homomorphism, then we have a cloning system on direct powers (G^n) of the group G and the Witzel-Zaremsky-Thompson group $\mathcal{T}(G^n)$ which was introduced by Tanushevski [6]. Suppose that G is left-orderable and fix a left-ordering $<$ of G . If the homomorphism $\phi_1: G \rightarrow G$ preserves the order of G , then the lexicographic orderings of G^n induced by $<$ are preserved by the cloning maps of (G^n) . Further if the order on G is bi-invariant, then the induced orders on G^n are also. Hence by Proposition 1.1 we have the following Theorem:

Theorem 4.5. *If the group G admits a left-ordering or bi-ordering preserved by the injective homomorphism $\phi_1: G \rightarrow G$, then the Witzel-Zaremsky-Thompson group $\mathcal{T}(G^n)$ of the direct powers of G is also left-orderable or bi-orderable, respectively.*

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